

# CFT Description of String Theory Compactified on Non-compact Manifolds with $G_2$ Holonomy

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*Dedicated to the memory of Sung-Kil Yang*

## Abstract

We construct modular invariant partition functions for strings propagating on non-compact manifolds of  $G_2$  holonomy. Our amplitudes involve a pair of  $\mathcal{N} = 1$  minimal models  $\mathcal{M}_m, \mathcal{M}_{m+2}$  ( $m = 3, 4, \dots$ ) and are identified as describing strings on manifolds of  $G_2$  holonomy associated with  $A_{m-2}$  type singularity. It turns out that due to theta function identities our amplitudes may be cast into a form which contain tricritical Ising model for any  $m$ . This is in accord with the results of Shatashvili and Vafa. We also construct a candidate partition function for string compactified on a non-compact Spin(7) manifold.

# 1 Introduction

Recently 7-dimensional manifolds with  $G_2$  holonomy are receiving a lot of attentions. These manifolds provide  $\mathcal{N} = 1$  4-dimensional compactifications of M-theory which are of fundamental physical interest. They also play an interesting role in a novel duality involving gauge and gravitational fields. In the duality conjectured by Vafa [1] Type IIA theory compactified on deformed conifold with D6-branes and Type IIA theory on resolved conifold with RR flux are related. This duality has been explained by lifting the Type IIA configurations to the M-theory backgrounds on three different manifolds of  $G_2$  holonomy which are smoothly connected to each other due to quantum effects [2, 3, 4]. This new type of duality has been further discussed in [5]-[13]. In the non-compact cases explicit Ricci flat metrics on manifolds of exceptional holonomy ( $G_2$  and  $\text{Spin}(7)$ ) have been known for some time [14, 15]. Recently new metrics have been discussed by various authors [16, 17, 18] while the existence theorems of metrics in the compact cases are given in [19].

In the case of string theory compactified on manifolds of exceptional holonomy we expect a more detailed description of the dynamics by making use of world-sheet techniques. World-sheet description of string theory on these manifolds was discussed by Shatashvili and Vafa [20]. In particular they argued that CFT description of manifolds with  $G_2$  holonomy contains the tricritical Ising model while that of 8-dimensional manifolds with  $\text{Spin}(7)$  holonomy involves the Ising model. We recall that the tricritical Ising model with central charge  $c = 7/10$  is the first one in the  $\mathcal{N} = 1$  Virasoro (unitary) minimal series (or the 2nd in the Virasoro  $\mathcal{N} = 0$  minimal series). Related results of CFT on manifolds with special holonomy are given in [21, 22].

In this article we construct modular invariant partition functions of strings compactified on non-compact manifolds with exceptional holonomies. We combine  $\mathcal{N} = 1$  Liouville field and  $\mathcal{N} = 1$  minimal models so that the states in NS and Ramond sectors cancel each other and the theory possesses space-time supersymmetry. We obtain a series of partition functions for strings on  $G_2$  holonomy manifolds associated with  $A_n$ -type singularities. We also obtain a partition function on a manifold with  $\text{Spin}(7)$  holonomy.

In the case of manifolds with  $G_2$  holonomy our amplitudes involve a pair of  $\mathcal{N} = 1$  minimal models  $\mathcal{M}_m^{\mathcal{N}=1}$ ,  $\mathcal{M}_{m+2}^{\mathcal{N}=1}$  for  $m = 3, 4, 5, \dots$ . It turns out that due to theta function identities our amplitudes may be rewritten into a form which contain tricritical Ising model together

with some  $\mathcal{N} = 0$  conformal system for any value of  $m$ . Thus our construction is consistent with the characterization of Shatashvili and Vafa [20] for manifolds with  $G_2$  holonomy. We can explicitly check the existence of the (unique) space-time supersymmetry operator in our partition functions. In the case of a non-compact manifold with  $\text{Spin}(7)$  holonomy we have a contribution of the superpartner of the Liouville field which is identified as the Ising model in the description of [20] of manifolds with  $\text{Spin}(7)$  holonomy.

## 2 Manifolds with $G_2$ Holonomy

We first consider the type II string theories compactified on the 7-dimensional manifolds with  $G_2$  holonomy. Our construction of partition function is analogous to that for singular Calabi-Yau manifolds [23, 24] which makes use of the  $\mathcal{N} = 2$  Liouville theory. Let us start with recalling briefly these constructions. Results of [23] may be reproduced starting from an identity

$$\chi_\ell^{(k)}(\tau) \left( \left( \frac{\theta_3(\tau)}{\eta(\tau)} \right)^4 - \left( \frac{\theta_4(\tau)}{\eta(\tau)} \right)^4 - \left( \frac{\theta_2(\tau)}{\eta(\tau)} \right)^4 \right) = 0, \quad \ell = 0, 1, 2, \dots, k. \quad (2.1)$$

Here  $\chi_\ell^{(k)}(\tau)$  denotes the character of the level- $k$   $SU(2)$  WZW model of spin  $\ell/2$  representation. In the above we have simply multiplied  $\chi_\ell^{(k)}(\tau)$  to the combination of theta functions which vanish due to the well-known Jacobi's identity. The factor  $\chi_\ell^{(k)}(\tau)$  is identified as describing the ALE space of  $A_{k+1}$  type. In fact if we extract a power of Jacobi theta function  $\theta_i$  ( $i = 1, 2, 3$ ) out of  $\theta_i^4$  and multiply it against  $\chi_\ell^k$ , we find

$$\sum_m \frac{\Theta_{m,k+2}}{\eta} \left( \left( \frac{\theta_3}{\eta} \right)^3 ch_{\ell,m}^{NS} - \left( \frac{\theta_4}{\eta} \right)^3 \tilde{ch}_{\ell,m}^{NS} - \left( \frac{\theta_2}{\eta} \right)^3 ch_{\ell,m}^R \right) = 0, \quad (2.2)$$

where  $ch_{\ell,m}^*$  denote the characters of  $\mathcal{N} = 2$  minimal model of level  $k$  and we have used the product formula of theta functions.  $\Theta_{m,k+2}$  denotes the standard theta function at level  $k+2$ . Eq.(2.2) is identified as describing the 6-dimensional string theory compactified on ALE space of the  $A_{k+1}$  type. Fermions in the four transverse directions in the Minkowski space  $\mathbf{R}^6$  together with the two fermions of  $\mathcal{N} = 2$  Liouville sector constitute the factor  $(\theta_i/\eta)^3$  in (2.2).  $U(1)$  theta function  $\Theta_{m,k+2}$  stands for the contribution of the momentum sum of the compact bosonic field  $Y$  in the  $\mathcal{N} = 2$  Liouville system [25, 23].

If one further extracts  $\theta_i$  out of  $\theta_i^3$ , multiplies it against  $\Theta_{m,k+2}$  and uses the product formula for theta functions, one obtains the conformal blocks of strings compactified on

Calabi-Yau 3-folds with the A-D-E singularities [23]. Construction for the Calabi-Yau 4-fold is similar.

Crucial ingredient in these constructions is the relationship between the character of  $SU(2)$  WZW model and the  $\mathcal{N} = 2$  minimal theory

$$\chi_\ell^{(k)} \times \theta_i \iff ch_{\ell,m}^* \times \Theta_{m,k+2}. \quad (2.3)$$

(2.3) lies behind the duality between the geometry of NS5-branes and ALE space [25].

Now, let us turn to the construction of partition functions for the case of  $G_2$  holonomy. In this case we have to use the  $\mathcal{N} = 1$  world-sheet SUSY rather than  $\mathcal{N} = 2$  SUSY. In the following we introduce an  $\mathcal{N} = 1$  Liouville system consisting of a scalar field  $\phi$  coupled to the background charge and a free Majorana fermion field  $\psi$  in order to describe non-compact space-time. Thus the total system consists of a CFT describing the geometry of the  $G_2$  manifold, the  $\mathcal{N} = 1$  Liouville theory and an additional free boson and fermion associated with the transverse direction of the Minkowski space  $\mathbf{R}^3$ .

We first recall that the  $SU(2) (SO(3))$  current algebra at level 2 may be realized by 3 Majorana fermions. Characters of the level 2  $SU(2)$  affine Lie algebra of spin  $\ell/2$  are in fact given by

$$\chi_{\ell=0}^{(2)} = \frac{1}{2} q^{-1/16} \left( \prod_{n=1}^{\infty} (1 + q^{n-1/2})^3 + \prod_{n=1}^{\infty} (1 - q^{n-1/2})^3 \right) = \frac{1}{2} \left\{ \left( \frac{\theta_3}{\eta} \right)^{3/2} + \left( \frac{\theta_4}{\eta} \right)^{3/2} \right\}, \quad (2.4)$$

$$\chi_{\ell=2}^{(2)} = \frac{1}{2} q^{-1/16} \left( \prod_{n=1}^{\infty} (1 + q^{n-1/2})^3 - \prod_{n=1}^{\infty} (1 - q^{n-1/2})^3 \right) = \frac{1}{2} \left\{ \left( \frac{\theta_3}{\eta} \right)^{3/2} - \left( \frac{\theta_4}{\eta} \right)^{3/2} \right\}, \quad (2.5)$$

$$\chi_{\ell=1}^{(2)} = 2q^{1/8} \prod_{n=1}^{\infty} (1 + q^n)^3 = \frac{1}{\sqrt{2}} \left( \frac{\theta_2}{\eta} \right)^{3/2}. \quad (2.6)$$

Thus roughly the  $3/2$  powers of the Jacobi theta functions give the level=2 characters and the factors  $\theta_i^4$  in (2.2) may be rewritten as

$$\theta_i^4 = \theta_i \times \chi_\ell^{(2)} \times \chi_{\ell'}^{(2)}. \quad (2.7)$$

We then recall the standard coset construction of  $\mathcal{N} = 1$  Virasoro minimal series given by

$$\mathcal{M}_m^{\mathcal{N}=1} : \frac{SU(2)_k \times SU(2)_2}{SU(2)_{k+2}}, \quad m = k + 2. \quad (2.8)$$

At the level of characters one has

$$\left( \frac{\theta_3}{\eta} \right)^{3/2} \chi_{r-1}^{(m-2)} = \sum_{\substack{s=1 \\ s-r \equiv 0 \pmod{2}}}^{m+1} \chi_{r,s}^{(m) NS} \chi_{s-1}^{(m)}, \quad (2.9)$$

$$\left(\frac{\theta_4}{\eta}\right)^{3/2} \chi_{r-1}^{(m-2)} = \sum_{\substack{s=1 \\ s-r \equiv 0 \pmod{2}}}^{m+1} \tilde{\chi}_{r,s}^{(m) NS} \chi_{s-1}^{(m)}, \quad (2.10)$$

$$\frac{1}{\sqrt{2}} \left(\frac{\theta_2}{\eta}\right)^{3/2} \chi_{r-1}^{(m-2)} = \sum_{\substack{s=1 \\ s-r \equiv 1 \pmod{2}}}^{m+1} \chi_{r,s}^{(m) R} \chi_{s-1}^{(m)}. \quad (2.11)$$

$\chi_{r,s}^{(m)*}$  denotes the characters of  $\mathcal{N} = 1$  minimal model of central charge  $c = 3/2(1 - 8/m(m+2))$ . See appendix for the explicit expressions of  $\mathcal{N} = 1$  characters (A.5), (A.7), (A.9).

Making use of (2.9), (2.10), (2.11) twice we can now rewrite (2.2) as

$$0 \equiv \chi_{r-1}^{(m-2)}(\tau) \times \frac{1}{\eta^4} (\theta_3^4 - \theta_4^4 - \theta_2^4) = 2 \sum_{s=1}^{m+3} F_{rs}^{(m)}(\tau) \chi_{s-1}^{(m+2)}(\tau). \quad (2.12)$$

Here the conformal blocks  $F_{r,s}^{(m)}$  are defined as

$$F_{rs}^{(m)}(\tau) \equiv \sum_{p=1}^{m+1} \left\{ \frac{1}{2} \frac{\theta_3}{\eta} \chi_{r,p}^{(m) NS} \chi_{p,s}^{(m+2) NS} - \frac{1}{2} \frac{\theta_4}{\eta} \tilde{\chi}_{r,p}^{(m) NS} \tilde{\chi}_{p,s}^{(m+2) NS} \right\} \\ - \sum_{p=1}^{m+1} \frac{\theta_2}{\eta} \chi_{r,p}^{(m) R} \chi_{p,s}^{(m+2) R}, \quad (2.13)$$

$$F_{m-r, m+4-s}^{(m)}(\tau) = F_{rs}^{(m)}(\tau), \quad (2.14)$$

where  $r, s$  run over the ranges  $1 \leq r \leq m-1$ ,  $1 \leq s \leq m+3$ , and  $r+s \equiv 0 \pmod{2}$ . Sum on  $p$  runs over  $r-p \equiv 0, s-p \equiv 0 \pmod{2}$  in NS sector while  $r-p \equiv 1, s-p \equiv 1 \pmod{2}$  in R sector.

Because of the identity (2.12) the branching functions  $F_{r,s}^{(m)}$  are expected to vanish for all values of  $m, r, s$ ,

$$F_{r,s}^{(m)} = 0, \quad \text{all } m, r, s. \quad (2.15)$$

We have explicitly verified this by Maple for smaller values of  $m$ .

Thus we have constructed the conformal blocks  $F_{r,s}^{(m)}$  for the candidate partition function for string theory on  $G_2$  holonomy manifolds. Blocks are made of a pair of  $\mathcal{N} = 1$  minimal models  $\mathcal{M}_m^{\mathcal{N}=1}, \mathcal{M}_{m+2}^{\mathcal{N}=1}$  with central charges  $c(m) = 3/2(1 - 8/m(m+2))$ ,  $c(m+2) = 3/2(1 - 8/(m+2)(m+4))$  and a single power of  $\theta_i$  being interpreted as the contribution of a transverse fermion in Minkowski 3-space and the fermion  $\psi$  of the  $\mathcal{N} = 1$  Liouville sector. Total central charge is given by

$$(1 + \frac{1}{2}) + (1 + 3Q^2 + \frac{1}{2}) + c(m) + c(m+2) = 12. \quad (2.16)$$

Here  $Q$  is the Liouville background charge which is adjusted to the value

$$Q = 2\sqrt{\frac{1}{2} + \frac{2}{m(m+4)}}. \quad (2.17)$$

As is obvious from the construction, blocks  $F_{r,s}^{(m)}$  have a good modular properties

$$F_{rs}^{(m)}\left(-\frac{1}{\tau}\right) = \sum_{r'=1}^{m-1} \sum_{\substack{s'=1 \\ |s'+r'|\equiv 0 \pmod{2}}}^{m+3} S_{rr'}^{(m-2)} S_{ss'}^{(m+2)} F_{r's'}^{(m)}(\tau), \quad (2.18)$$

$$F_{rs}^{(m)}(\tau+1) = \exp\left\{2\pi i \left(\frac{1}{3} + \frac{r^2}{4m} - \frac{s^2}{4(m+4)}\right)\right\} F_{rs}^{(m)}(\tau). \quad (2.19)$$

Here  $S_{rr'}^{(k)} \equiv \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi r r'}{k+2}\right)$  denotes the modular matrix of  $SU(2)_k$ . The following combination of conformal blocks gives a modular invariant partition function:

$$Z(\tau, \bar{\tau}) = \sum_{r, \bar{r}=1}^{m-1} \sum_{s, \bar{s}=1}^{m+3} Z_0(\tau, \bar{\tau}) (N_{r, \bar{r}}^{(m-2)} N_{s, \bar{s}}^{(m+2)} + N_{r, m-\bar{r}}^{(m-2)} N_{s, m+4-\bar{s}}^{(m+2)}) F_{rs}^{(m)}(\tau) \overline{F_{\bar{r}\bar{s}}^{(m)}(\bar{\tau})}, \quad (2.20)$$

$$s + r \equiv \bar{s} + \bar{r} \equiv 0 \pmod{2}.$$

Here we may use any coefficient set of modular invariants  $N_{r, \bar{r}}^{(m-2)}$ ,  $N_{s, \bar{s}}^{(m+2)}$  of  $SU(2)_{m-2}$ ,  $SU(2)_{m+2}$  theories.  $Z_0$  denotes the trivial part of the partition function which does not enter into the GSO projection

$$Z_0 = \frac{1}{|\prod_{n=1}^{\infty} (1 - q^n)|^4} \int_{-\infty}^{+\infty} dp dp_L \exp\left(-4\pi\tau_2 \left(\frac{1}{2}p^2 + \frac{1}{2}p_L^2 + \frac{1}{8}Q^2 - \frac{(c_L+1)}{24}\right)\right) \\ = \frac{1}{\tau_2 |\eta(\tau)|^4}, \quad (2.21)$$

where  $c_L$  denotes the Liouville central charge  $c_L = 1+3Q^2$  and  $\tau_2 = \text{Im } \tau$ .  $p_L(p)$  is the Liouville (Minkowski) momentum. As is well-known, Liouville spectrum has a gap  $h(L) \geq Q^2/8$ .

We note that there exists a unique operator (in each chiral sector) which generates the analogue of the spectral flow in  $\mathcal{N} = 2$  theories between NS and Ramond sectors and is identified as the space-time SUSY operator in the above partition function. In fact the operator  $\Psi = \phi_{1,2}^{(m)} \phi_{2,1}^{(m+2)}$  contained in R sector has a dimension  $h_{1,2}^{(m)} + h_{2,1}^{(m+2)} = 3/8$  for any values of  $m$ . When it is properly dressed by the superconformal ghost and spin fields, it gives a current of conformal dimension 1

$$J_{L,R} = e^{-\phi_{gh}/2} S_\alpha \Psi_{L,R}, \quad (2.22)$$

(spin field  $S_\alpha$  contains the contribution from the fermion of  $\mathcal{N} = 1$  Liouville theory and has dimension  $2/8$ ).

Dimension of the fields  $\phi_{r,p}^{(m)}\phi_{p,s}^{(m+2)}$  which appear in the block  $F_{r,s}^{(m)}$  is in general given by

$$h_p^{r,s} = \frac{1}{4} \left( p - \frac{r+s}{2} \right)^2 + \frac{((m+4)r - ms)^2 - 16}{16m(m+4)} + \frac{\epsilon}{8}, \quad (2.23)$$

where  $\epsilon = 0$  for  $p+r, p+s \equiv 0 \pmod{2}$  in NS sector and  $\epsilon = 1$  for  $p+r, p+s \equiv 1 \pmod{2}$  in R sector. Thus in the “graviton orbit”  $r = s = 1$ ,

$$h_p^{1,1} = \frac{1}{4}(p-1)^2 + \frac{\epsilon}{8}, \quad p \leq m+1. \quad (2.24)$$

Hence fields in the NS sector of graviton orbit all possess integer conformal dimensions. This suggests the existence of an extension of the chiral algebra to some algebra involving higher spin fields in our construction. In fact  $h_p^{r,s}$  and  $h_{p\pm 2}^{r,s}$  differ by integers for any  $p, r, s$  and it seems quite likely that the sum over the product of minimal characters  $\sum_p \chi_{r,p}^{(m)*} \chi_{p,s}^{(m+2)*}$  provides a character of an irreducible representation  $(r, s)$  of the extended algebra. Such an extended algebra for manifolds with exceptional holonomy was first introduced by Shatashvili and Vafa [20] and further studied in refs.[21, 22]. We also note the pairing of NS and Ramond states

$$h_{p+1}^{r,s} - h_p^{r,s} = \frac{3}{8} + \frac{1}{2} + \text{integer}, \quad p = \text{odd}. \quad (2.25)$$

The dimension  $3/8$  is compensated by the spectral flow operator  $\Psi$  and  $1/2$  is consistent with the GSO condition for NS sector incorporated in the conformal blocks (2.13). If we recall the OPE of the minimal model

$$\phi_{r,p}\phi_{1,2} \approx \phi_{r,p\pm 1}, \quad (2.26)$$

we note that the operator  $\Psi$  in fact generates a spectral flow. We identify the state  $r = s = 1, p = 3$  in the graviton orbit as the associative 3-form  $\Phi$  of the  $G_2$  holonomy manifold since it has dimension  $3/2$  (contribution from the fermions is added) and acts like the square of the spectral flow operator.

We also note that due to the presence of the gap in Liouville spectrum, the dimension of the Ramond ground state satisfies an inequality  $h(m) + h(m+2) + h(L) \geq 1/24(c(m) + c(m+2) + c(L)) = 3/8$ . Together with the contribution from  $\theta_2$  it adds up to  $1/2$  which is the value dictated by space-time SUSY. Thus we believe that the partition function (2.20) satisfies all the necessary conditions for the string theory compactified on non-compact  $G_2$  holonomy manifolds.

Except the special case of  $m = 3$  (2.13) does not appear to contain tricritical Ising model. It turns out, however, due to some theta function identities we can recast (2.13) into a form which explicitly contain tricritical Ising model together with some  $\mathcal{N} = 0$  rational conformal models.<sup>1</sup> Explicit expressions are presented in section 4.

### 3 Manifold with Spin(7) Holonomy

Next we would like to discuss a modular invariant for a non-compact 8-dimensional manifold with Spin(7) holonomy. Now we start from the following identities

$$f_1(\tau) \equiv \frac{1}{2} \left\{ \left( \frac{\theta_3}{\eta} \right)^2 - \left( \frac{\theta_4}{\eta} \right)^2 \right\} \chi_0^{(1)}(\tau) - \frac{1}{2} \left( \frac{\theta_2}{\eta} \right)^2 \chi_1^{(1)}(\tau) = 0, \quad (3.1)$$

$$f_2(\tau) \equiv \frac{1}{2} \left\{ \left( \frac{\theta_3}{\eta} \right)^2 + \left( \frac{\theta_4}{\eta} \right)^2 \right\} \chi_1^{(1)}(\tau) - \frac{1}{2} \left( \frac{\theta_2}{\eta} \right)^2 \chi_0^{(1)}(\tau) = 0. \quad (3.2)$$

Here  $\chi_\ell^{(1)}$  is the character of level 1  $SU(2)$  WZW model with spin  $\ell/2$  representation. (3.1),(3.2) are related to the Jacobi's identity as

$$\frac{1}{2} \left( \left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right) = \chi_0^{(1)} f_1(\tau) + \chi_1^{(1)} f_2(\tau) \quad (3.3)$$

and were used previously in [26, 24, 23].

In the present context we first decompose  $\theta_i^2$  as  $\theta_i^{1/2} \times \theta_i^{3/2}$  and replace  $\theta_i^{3/2}$  by the level 2  $SU(2)$  characters (2.9),(2.10),(2.11). We then find the product of level 1 and 2  $SU(2)$  characters which gives rise to the tricritical Ising model under the standard coset construction of  $\mathcal{N} = 0$  minimal series

$$\mathcal{M}_m^{\mathcal{N}=0} : \frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}, \quad m = k + 2. \quad (3.4)$$

Tricritical Ising model is identified as  $\mathcal{M}_{m=4}^{\mathcal{N}=0}$  and has conformal blocks with dimensions

$$[h = 0], \quad [h = \frac{1}{10}], \quad [h = \frac{3}{5}], \quad [h = \frac{3}{2}] \quad (3.5)$$

$$[h = \frac{3}{80}], \quad [h = \frac{7}{16}]. \quad (3.6)$$

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<sup>1</sup>We are grateful to C.Vafa for insisting on the existence of the tricritical Ising model in the case of non-compact manifolds of  $G_2$  holonomy. This prompted us to look for a formula involving tricritical Ising model and correct an error in the original version of the manuscript.



It is known that tricritical Ising model possesses an  $\mathcal{N} = 1$  SUSY and is also identified as the first one  $\mathcal{M}_{m=3}^{\mathcal{N}=1}$  in the minimal  $\mathcal{N} = 1$  series. In  $\mathcal{N} = 1$  terms conformal blocks are organized as

$$\text{NS sector : } [h = 0], [h = \frac{1}{10}], \quad (3.7)$$

$$\text{R sector : } [h = \frac{3}{80}], [h = \frac{7}{16}]. \quad (3.8)$$

On the other hand the square roots of Jacobi theta functions may be identified as characters of the Ising model

$$\chi_{h=0}^{\text{Is}} = \frac{1}{2} \left( \sqrt{\frac{\theta_3}{\eta}} + \sqrt{\frac{\theta_4}{\eta}} \right) \quad (3.9)$$

$$\chi_{h=1/2}^{\text{Is}} = \frac{1}{2} \left( \sqrt{\frac{\theta_3}{\eta}} - \sqrt{\frac{\theta_4}{\eta}} \right) \quad (3.10)$$

$$\chi_{h=1/16}^{\text{Is}} = \frac{1}{\sqrt{2}} \sqrt{\frac{\theta_2}{\eta}} \quad (3.11)$$

Let us now introduce conformal blocks for the candidate partition function of string compactified on a non-compact manifold of Spin(7) holonomy

$$F_1(\tau) \equiv \chi_{h=0}^{\text{Is}} \chi_{h=3/2}^{\text{tri}} + \chi_{h=1/2}^{\text{Is}} \chi_{h=0}^{\text{tri}} - \chi_{h=1/16}^{\text{Is}} \chi_{h=7/16}^{\text{tri}}, \quad (3.12)$$

$$F_2(\tau) \equiv \chi_{h=0}^{\text{Is}} \chi_{h=1/10}^{\text{tri}} + \chi_{h=1/2}^{\text{Is}} \chi_{h=3/5}^{\text{tri}} - \chi_{h=1/16}^{\text{Is}} \chi_{h=3/80}^{\text{tri}}. \quad (3.13)$$

Identities (3.1), (3.2) are then expressed as

$$f_1(\tau) = F_1(\tau) \chi_0^{(3)}(\tau) + F_2(\tau) \chi_2^{(3)}(\tau) = 0, \quad (3.14)$$

$$f_2(\tau) = F_1(\tau) \chi_3^{(3)}(\tau) + F_2(\tau) \chi_1^{(3)}(\tau) = 0. \quad (3.15)$$

Thus the branching functions  $F_i$  ( $i = 1, 2$ ) in fact vanish

$$F_i = 0, \quad i = 1, 2 \quad (3.16)$$

consistently with the space-time SUSY. We have checked (3.16) by Maple. By construction blocks  $F_i$  have good modular properties. It is easy to derive the S-transformation of the conformal blocks

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \left( -\frac{1}{\tau} \right) = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\left(\frac{\pi}{5}\right) & \sin\left(\frac{2\pi}{5}\right) \\ \sin\left(\frac{2\pi}{5}\right) & -\sin\left(\frac{\pi}{5}\right) \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}(\tau) \quad (3.17)$$

In the present case of 8-dimensional manifold there are no transverse degrees of freedom of Minkowski space. We interpret the Ising model sector as arising from the Liouville fermion  $\psi$ . We note that there exists a space-time SUSY operator in the partition function: a primary field  $h = 7/16$  in Ramond sector together with the superconformal ghost and spin field gives a current  $h = 1$  (spin field contains a contribution from the Liouville fermion and has a dimension  $3/16$ ).

Taking account of the contribution of the Liouville field the partition function is then given by

$$Z = \frac{1}{\tau_2^{1/2} |\eta(\tau)|^2} \left( |F_1(\tau)|^2 + |F_2(\tau)|^2 \right) \quad (3.18)$$

## 4 Discussions

In this article we have constructed candidate partition functions of string theory compactified on non-compact manifolds with exceptional holonomy. We have combined Liouville and  $\mathcal{N} = 1$  minimal models so that the states in NS and Ramond sectors cancel and the theory possesses space-time supersymmetry. It seems quite likely that the manifolds of  $G_2$  holonomy described by our construction are the ALE spaces of  $A_n$  type fibered over  $S^3$ . The existence of  $SU(2)$  current algebra originating from the  $SU(2)$  holonomy of ALE spaces is crucial in our construction. We have found that pairs of  $\mathcal{N} = 1$  minimal models enter into the partition function which contain spectral flow and SUSY generators. Our construction is relatively easy since one can freely adjust the central charge by Liouville field.

Let us now check that our amplitudes  $F_{rs}^{(m)}$  (2.13) in fact contain tricritical Ising model as discussed in [20]. First we combine relations (3.3), (3.14), (3.15) and obtain

$$\begin{aligned} \chi_{r-1}^{(m-2)} \frac{1}{\eta^4} (\theta_3^4 - \theta_4^4 - \theta_2^4) &= 2\chi_{r-1}^{(m-2)} (\chi_0^{(1)} \chi_0^{(3)} + \chi_1^{(1)} \chi_3^{(3)}) F_1 \\ &\quad + 2\chi_{r-1}^{(m-2)} (\chi_0^{(1)} \chi_2^{(3)} + \chi_1^{(1)} \chi_1^{(3)}) F_2. \end{aligned} \quad (4.1)$$

We then introduce the branching functions  $\chi_{(r,s;t)}^{(M;L)}(\tau)$  for the cosets  $SU(2)_M \times SU(2)_L / SU(2)_{M+L}$ ,

$$\chi_{r-1}^{(M)}(\tau) \chi_t^{(L)}(\tau) = \sum_{\substack{s=1 \\ |r-s-t| \equiv 0 \pmod{2}}}^{M+L+1} \chi_{(r,s;t)}^{(M;L)}(\tau) \chi_{s-1}^{(M+L)}(\tau), \quad (4.2)$$

$$(1 \leq r \leq M+1, 1 \leq s \leq M+L+1, 0 \leq t \leq L),$$

$$\chi_{(r,s;t)}^{(M;L)} = \chi_{(M-r+2, M+L-s+2; L-t)}^{(M;L)}, \quad (4.3)$$

and obtain

$$\begin{aligned} \chi_{r-1}^{(m-2)} \frac{1}{\eta^4} (\theta_3^4 - \theta_4^4 - \theta_2^4) &= 2 \sum_{s=1}^{m+3} \chi_{s-1}^{(m+2)}(\tau) \left[ F_1(\tau) \left\{ \sum_p \chi_{(r,p;0)}^{(m-2;1)} \chi_{(p,s;0)}^{(m-1;3)} + \sum_{p'} \chi_{(m-r,p';0)}^{(m-2;1)} \chi_{(p',m+4-s;0)}^{(m-1;3)} \right\}(\tau) \right. \\ &\quad \left. + F_2(\tau) \left\{ \sum_p \chi_{(r,p;0)}^{(m-2;1)} \chi_{(p,s;2)}^{(m-1;3)} + \sum_{p'} \chi_{(m-r,p';0)}^{(m-2;1)} \chi_{(p',m+4-s;2)}^{(m-1;3)} \right\}(\tau) \right]. \end{aligned} \quad (4.4)$$

By comparing with (2.12) we find

$$\begin{aligned} F_{rs}^{(m)}(\tau) &= F_1(\tau) \left\{ \sum_p \chi_{(r,p;0)}^{(m-2;1)} \chi_{(p,s;0)}^{(m-1;3)} + \sum_{p'} \chi_{(m-r,p';0)}^{(m-2;1)} \chi_{(p',m+4-s;0)}^{(m-1;3)} \right\}(\tau) \\ &\quad + F_2(\tau) \left\{ \sum_p \chi_{(r,p;0)}^{(m-2;1)} \chi_{(p,s;2)}^{(m-1;3)} + \sum_{p'} \chi_{(m-r,p';0)}^{(m-2;1)} \chi_{(p',m+4-s;2)}^{(m-1;3)} \right\}(\tau). \end{aligned} \quad (4.5)$$

Thus our conformal blocks are now expressed in terms of tricritical Ising model together with some combinations of  $\mathcal{N} = 0$  rational conformal field theories. This description seems to fit exactly with the discussion given by Shatashvili and Vafa [20]. In the right-hand-side of (4.5)  $\mathcal{N} = 1$  world-sheet SUSY is not manifest and also the Liouville fermion is not explicit. In our original representation (2.13), on the other hand, world-sheet SUSY and the Liouville fermion are manifest but the tricritical Ising model is difficult to identify.

In the case of compact manifolds one can no longer use the Liouville field and can not freely adjust the central charge of the theory: the system is much more rigid. Construction of modular invariants for compact manifolds is a challenging problem which we would like to address in future communications.

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# Appendix: $\mathcal{N} = 1$ Unitary Minimal Models

We summarize basic data on  $\mathcal{N} = 1$  Virasoro minimal models.

1. Central charge and conformal dimensions:

$$c = \frac{3}{2} - \frac{12}{m(m+2)}, \quad (m = 3, 4, 5, \dots) \quad (\text{A.1})$$

$$\begin{aligned} h_{r,s}^{(m)} &= \frac{((m+2)r - ms)^2 - 4}{8m(m+2)} + \frac{1 - (-1)^{r-s}}{32}, \\ &\equiv \frac{((m+2)r - ms)^2 - 4}{8m(m+2)} + \frac{\epsilon}{16} \end{aligned} \quad (\text{A.2})$$

$$\epsilon = \begin{cases} 0 & r + s \equiv 0 \pmod{2} : \text{ NS-sector} \\ 1 & r + s \equiv 1 \pmod{2} : \text{ R-sector} \end{cases} \quad (\text{A.3})$$

$$1 \leq r \leq m-1, \quad 1 \leq s \leq m+1$$

$$h_{m-r, m+2-s}^{(m)} = h_{r,s}^{(m)}. \quad (\text{A.4})$$

2. Character formulas [27]:

$$\chi_{r,s}^{(m) NS}(\tau) = \frac{1}{\eta} \sqrt{\frac{\theta_3}{\eta}} K_{r,s}^{(m)}(\tau) \equiv q^{-\frac{1}{16}} \prod_{n=1}^{\infty} \frac{1 + q^{n-\frac{1}{2}}}{1 - q^n} K_{r,s}^{(m)}(\tau), \quad (\text{A.5})$$

$$\begin{aligned} K_{r,s}^{(m)}(\tau) &\stackrel{\text{def}}{=} \Theta_{(m+2)r-ms, 2m(m+2)}(\tau) + \Theta_{(m+2)r-ms+2m(m+2), 2m(m+2)}(\tau) \\ &\quad - \Theta_{(m+2)r+ms, 2m(m+2)}(\tau) - \Theta_{(m+2)r+ms+2m(m+2), 2m(m+2)}(\tau) \\ &= \Theta_{(m+2)r-ms, m(m+2)}(\tau/2) - \Theta_{(m+2)r+ms, m(m+2)}(\tau/2). \end{aligned} \quad (\text{A.6})$$

$$\tilde{\chi}_{r,s}^{(m) NS}(\tau) = \frac{1}{\eta} \sqrt{\frac{\theta_4}{\eta}} \tilde{K}_{r,s}^{(m)}(\tau) \equiv q^{-\frac{1}{16}} \prod_{n=1}^{\infty} \frac{1 - q^{n-\frac{1}{2}}}{1 - q^n} \tilde{K}_{r,s}^{(m)}(\tau), \quad (\text{A.7})$$

$$\begin{aligned} \tilde{K}_{r,s}^{(m)}(\tau) &\stackrel{\text{def}}{=} (-1)^{\frac{r-s}{2}} \Theta_{(m+2)r-ms, 2m(m+2)}(\tau) \\ &\quad + (-1)^{\frac{r-s}{2}+m} \Theta_{(m+2)r-ms+2m(m+2), 2m(m+2)}(\tau) \\ &\quad - (-1)^{\frac{r+s}{2}} \Theta_{(m+2)r+ms, 2m(m+2)}(\tau) \\ &\quad - (-1)^{\frac{r+s}{2}+m} \Theta_{(m+2)r+ms+2m(m+2), 2m(m+2)}(\tau). \end{aligned} \quad (\text{A.8})$$

$$\chi_{r,s}^{(m) R}(\tau) = \frac{1}{\sqrt{2}\eta} \sqrt{\frac{\theta_2}{\eta}} K_{r,s}^{(m)}(\tau) \equiv \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} K_{r,s}^{(m)}(\tau) \quad (\text{A.9})$$

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